Jan 12, 2017

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Recall that in order to study limit, convergence, approximation, continuity, ... we do not need a metric, instead only topology $\mathcal{J} \subset \mathcal{P}(X)$. It is a system satisfying

(T1) $\forall \mathcal{G} \subset \mathcal{J}, \quad \bigcup \mathcal{G} \in \mathcal{J}$

(T2) $\forall G_1, \cdots, G_n \in \mathcal{J}, \quad \bigcap_{k=1}^{n} G_k \in \mathcal{J}$

Terminology. We say that $\mathcal{J}$ is closed under arbitrary union and finite intersection.

An element in $\mathcal{J}$ is an open set.

Examples.

* Let $X = \mathbb{R}$ and

$$\mathcal{J} = \{\phi, \mathbb{R}\} \cup \{(a-\varepsilon, a+\varepsilon),\ a \in \mathbb{R},\ \varepsilon > 0\}$$

This is not a topology as (T1) is not valid.

* Let $X = \mathbb{R}$

$$\mathcal{J} = \{\phi, \mathbb{R}, [1,3], [2,4], [1,4], [2,3]\}$$

Is $[1,3]$ open?

Yes, because $\mathcal{J}$ is a topology (T1 & T2)

Notation. A topological space $(X, \mathcal{T})$

Recall Discrete topology $\mathcal{T} = \mathcal{P}(X)$ comes from a metric.

What about Indiscrete Topology $\mathcal{T} = \{\phi, X\}$

We cannot test metrics one by one. Let us observe what happens under a metric d.

Let $x \neq y \in X$, with the metric, we have

$$d(x,y) = r > 0$$

Take the ball $B(x, \frac{r}{3}) = \{ z \in X : d(z,x) < \frac{r}{3} \}$ and $B(y, \frac{r}{3})$

We have $x \in B(x, \frac{r}{3})$ and $y \in B(y, \frac{r}{3})$

but $B(x, \frac{r}{3}) \cap B(y, \frac{r}{3}) = \phi$

need Δ-inequality

Here is a special property

$$\boxed{\begin{array}{l} \text{For each pair } x \neq y \in X,\ \exists\, U, V \in \mathcal{T} \\ x \in U,\ y \in V,\ U \cap V = \phi \end{array}}$$

It is called Hausdorff or T_2

Fact. A metric space is Hausdorff

For $\mathcal{T} = \{\phi, X\}$ with $\#X \geq 2$, it is not T_2 and thus not comes from a metric

Is cofinite topology Hausdorff ?

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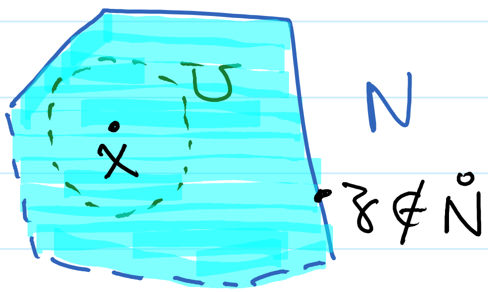
How to define interior?

Given $(X, \mathcal{T})$ and $N \subset X$

A point $x \in N$ is an **interior point** of N

if $\exists\, U \in \mathcal{T}$ $x \in U \subset N$



We also say that

N is a **neighborhood** of x

Notation. $x \in \mathring{N}$ or $x \in \mathrm{Int}(N)$

The set containing

all interior points of N

Quick Facts

* $A \in \mathcal{T} \iff A = \mathring{A} \iff A \subset \mathring{A}$
* $\mathring{A}$ is the largest open set contained in A

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Given $(X, \mathcal{T})$, for each $x \in X$

define $\mathcal{N}_x = \{N \subset X : N \text{ is a nbhd of } x\}$

i.e. $x \in \mathring{N}$

Warn: N may not be open

The collection $\mathcal{N}_x$ satisfies

(N1) $\forall N \in \mathcal{N}_x$, $x \in N$ Obvious

(N2) $\forall M, N \in \mathcal{N}_x$ $M \cap N \in \mathcal{N}_x$ trivial

(N3) If $N \in \mathcal{N}_x$ and $M \supset N$ trivial
then $M \in \mathcal{N}_x$

(N4) For $N \in \mathcal{N}_x$, temporarily let
$\overset{\text{eee}}{N} = \{y \in N : N \in \mathcal{N}_y\}$ simple
then $\overset{\text{eee}}{N} \in \mathcal{N}_x$

Terminology $\{\mathcal{N}_x : x \in X\}$ satisfying
(N1) — (N4) is a neighborhood system for X

Fact. A nbhd system determines a
unique topology of X such that ......

Think $X = \mathbb{R}$ $\mathcal{U}_x = \{(x - \frac{1}{n}, x + \frac{1}{n}) : 1 \leq n \in \mathbb{Z}\}$